Designing Averaging Filters for Mild Lowpass Filtering and Removal of Harmonics

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In data analysis, e.g. number of new confirmed COVID-19 cases in Texas each day as shown on the right, we'll often see a moving average to help us discern an underlying trend. The trend in certain segments of the data might show a decrease or increase in what's being measured. A moving average averages out, or smoothes out, rapid fluctuations in the data. This is equivalent to keeping low frequency information and attenuating high frequency information in the data.

Number of new confirmed COVID-19 cases in The State of Texas as of Feb. 7, 2021, reported by The State of Texas Department of State Health Services.

For a data sequence $x[n]$, a seven-sample moving average can be computed as follows:

$$y[n] = \frac{1}{7}(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6])$$

for $n \geq 0$. The output $y[n]$ is the average of the current data point $x[n]$ and the six previous data points. This difference equation is also known as a seven-coefficient averaging filter.

Analysis. We analyze the frequency response for a linear time-invariant (LTI) system using the discrete-time Fourier transform. A necessary condition for a system to be LTI is that the system must initially be "at rest". In this case, it means that the initial conditions must be zero. We can identify the initial conditions in several ways. One way is to start computing output values:

$$y[0] = \frac{1}{7}(x[0] + x[-1] + x[-2] + x[-3] + x[-4] + x[-5] + x[-6])$$

The six initial conditions correspond to the six memory locations storing the previous six input values. When $n = 0$, there are no previous input values. We initialize the six memory locations $\{ x[n-1], x[n-2], x[n-3], x[n-4], x[n-5], x[n-6] \}$ to zero to make the system be initially at rest. Initializing data variables and arrays to zero is a common programming practice.

There are many ways to find the discrete-time Fourier transform of the seven-coefficient LTI averaging filter. One way is to find the impulse response and then find its discrete-time Fourier transform. The impulse response is the response of a system to a discrete-time impulse:
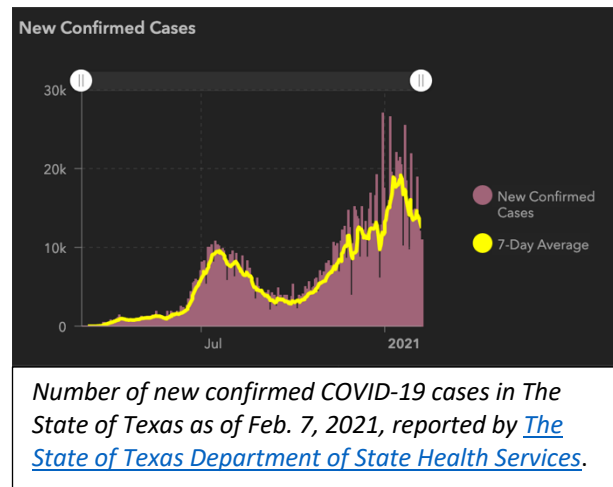
$$h[n] = \frac{1}{7}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6])$$

The impulse response $h[n]$ is a rectangular pulse that is seven samples long with amplitude 1/7 as shown on the next page. We can find the Fourier transform from the z-transform:

$$H(z) = \frac{1}{7}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}) \ \ \text{for } z \neq 0$$

Since the region of convergence $z \neq 0$ includes the unit circle, we can substitute $z = e^{j\omega}$:

$$H_{freq}(\omega) = H(e^{j\omega}) = \frac{1}{7}(1 + e^{-j\omega} + e^{-2j\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega} + e^{-j6\omega})$$

We can factor out the middle term (due to the phase shift corresponding to the mid-point of the impulse response at sample index $(L-1)/2$) to reveal that the phase is linear in frequency:

$$H_{freq}(\omega) = \frac{1}{7}e^{-j3\omega}\left(e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}\right)$$
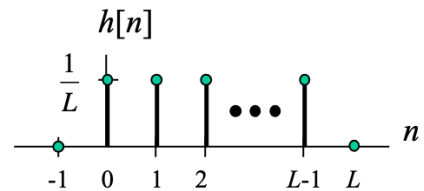
$$H_{freq}(\omega) = \frac{1}{7}(1 + 2\cos(\omega) + 2\cos(2\omega) + 2\cos(3\omega))\, e^{-j3\omega} = A(\omega)\, e^{-j3\omega}$$

Amplitude $A(\omega)$ is a sinc with period 2π and the phase is -3ω which is a line with slope of -3. Please note that $A(\omega)$ has positive, zero, and negative values, and is not the magnitude.
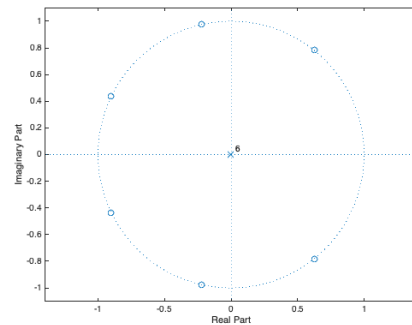
The impulse response (on right) is a rectangular pulse of 7 samples with amplitude 1/7. The pole-zero plot for H(z) shows zeros at angles (frequencies) that are multiples of $2\pi/7$ rad/sample except at zero. Magnitude response is lowpass with a null bandwidth of $2\pi/7$ rad/sample and frequencies at multiples of $2\pi/7$ rad/sample are zeroed out except at 0 rad/sample. Phase response is linear except for the discontinuities at frequencies that are zeroed out (not passed) by the filter at multiples of $2\pi/7$ rad/sample except at 0 rad/sample.

$h[n]$

Impulse response for LTI averaging filter with L coefficients (L = 7 here)

Design a lowpass discrete-time LTI filter with a null bandwidth of 60 Hz that eliminates multiples of the powerline frequency 60 Hz up to half of the sampling rate. The sampling rate is 420 Hz.

Answer: The null bandwidth in discrete-time frequency is $\omega_0 = 2\pi\frac{f_0}{f_s} = 2\pi\frac{60\,Hz}{420\,Hz} = \frac{2}{7}\pi$ rad/sample. The null bandwidth for an LTI averaging filter with L coefficients is $2\pi/L$; hence, $L = 7$. Averaging filter with 7 coefficients will eliminate 60 Hz, 120 Hz, and 180 Hz for a sampling rate of 420 Hz.
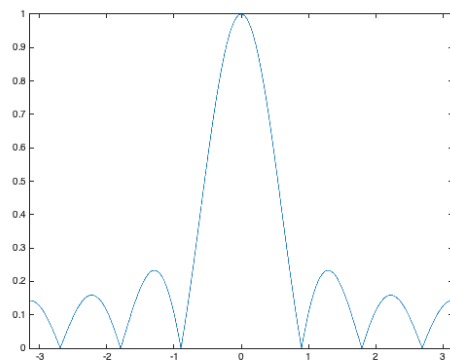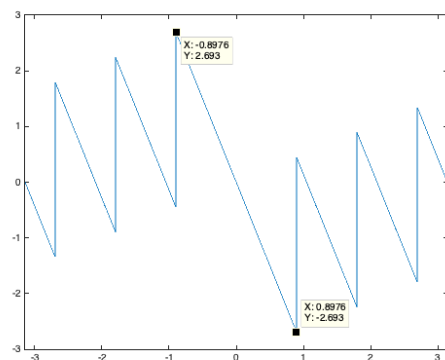

Poles & zeros of H(z) for LTI averaging filter with 7 coefficients in Matlab:

```
h = (1/7)*ones(1, 7); zplane(h)
```


Magnitude response for LTI averaging filter with 7 coefficients. Magnitude of a periodic sinc. Zeros at multiples of $2\pi/7$ except 0.

Phase response for LTI averaging filter with 7 coefficients in Matlab. Using the highlighted points, the slope is estimated to be -3.0002.

```
w = -pi : (2*pi/100000) : pi;
H = (1/7)*(1 + exp(-j*w) + exp(-j*2*w) + exp(-j*3*w) + exp(-j*4*w) + exp(-j*5*w) + exp(-j*6*w));
figure; plot(w, abs(H)); xlim( [-pi, pi] );    %%% plot the magnitude response
figure; plot(w, phase(H)); xlim( [-pi, pi] );   %%% plot the phase response
```